Extending the ADM formalism to Weyl geometry

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Abstract

In order to treat quantum cosmology in the framework of Weyl spacetimes we take the first step of extending the Arnowitt-Deser-Misner formalism to Weyl geometry. We then obtain an expression of the curvature tensor in terms of spatial quantities by splitting spacetime in (3+1)-dimensional form. We next write the Lagrangian of the gravitation field based in Weyl-type gravity theory. We extend the general relativistic formalism in such a way that it can be applied to investigate the quantum cosmology of models whose spacetimes are endowed with a Weyl geometrical structure.

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I. INTRODUCTION

"*Quantum cosmology is the application of quantum theory to the dynamical systems describing closed cosmology.*" (J.J. Halliwell)[1]

It is with the sentence above that we would like to begin this work, because it appropriately summarizes the approach to quantum cosmology. Another sentence which we could quote to justify, in part, our interest in the Weyl geometry follows from Dirac's words:

"*It appears as one of the fundamental principles of nature that the equations expressing the basic laws of physics should be invariant under the widest possible group of transformations.*" (P.A.M. Dirac)[2]

In this work, we consider a gravity theory formulated in the language of Weyl geometry. Our aim is to develop a formalism that can further be applied to quantum cosmology. In this geometric structure it is given a purely geometric scalar field that could play a role to address some issues, such as the *time problem*[3, 4]. Clearly the first step to carry out this program is to extend the Arnowitt-Deser-Misner formalism (ADM formalism)[5] from Riemannian to Weyl geometry.

In this paper, we briefly review the main points of the Weyl geometry, in particular those related to the so-called Weyl integrable geometries. Furthermore, we apply the formalism to some examples in order to find the role of the Weyl field as a canonical variable of the system. [6]

II. WEYL GEOMETRY

The geometry conceived by Weyl is a simple generalization of Riemannian geometry[7, 8]. Instead of postulating that the covariant derivative of the metric tensor g is zero, we assume the more general condition

$$\nabla_\alpha g_{\mu\nu} = \sigma_\alpha g_{\mu\nu}, \tag{1}$$

where σ_α denotes the components, with respect to a local coordinates basis $\{\partial_\alpha\}$, of a 1-form field σ defined on the manifold $\mathcal{M}$. This represents a generalization of the Riemannian condition of compatibility between the connection ∇ and the metric tensor g, namely, the requirement that the length of a vector remains unaltered by parallel transport. If σ vanishes,

then (1) reduces to the familiar Riemannian metricity condition. It is noteworthy that the Weyl condition (1) remains unchanged when we perform the following simultaneous transformations in g and σ:

$$\bar{g} = \mathrm{e}^{f} g, \tag{2}$$

$$\bar{\sigma} = \sigma + df, \tag{3}$$

where f is an arbitrary scalar function defined on $\mathcal{M}$. These are known in the literature as Weyl transformations.

If $\sigma = d\phi$, where ϕ is a scalar field, then we have what is called a *Weyl integrable manifold.* In this particular case, the transformations (2) and (3) become

$$\bar{g} = \mathrm{e}^{f} g, \tag{4}$$

$$\bar{\phi} = \phi + f. \tag{5}$$

In this paper, we consider only Weyl integrable manifolds.

If the connection ∇ is assumed to be torsionless, then by virtue of condition (1) it gets completely determined by g and ϕ. Indeed, a straightforward computation shows that the components of the affine connection with respect to an arbitrary vector basis are completely given by

$$\Gamma^{\alpha}_{\mu\nu} = \left\{ {}^{\alpha}_{\mu\nu} \right\} - \frac{1}{2} g^{\alpha\beta} \left(g_{\beta\mu}\phi_{\nu} + g_{\beta\nu}\phi_{\mu} - g_{\mu\nu}\phi_{\beta} \right), \tag{6}$$

where $\left\{ {}^{\alpha}_{\mu\nu} \right\}$ represents the Christoffel symbols and we are denoting $\phi_{\mu} \doteq \partial_{\mu}\phi$. The affine connection given in (6) is called *Weyl connection* and it is invariant under the Weyl transformations. A nice account of Weyl's ideas, as well as the refutation of his gravitational theory, may be found in reference[9].

III. THE SPLITTING OF THE SPACETIME

As is well known, the first obstacle that we have to deal with in the quantum cosmology approach is the incompatibility between geometrical gravity theories and the Hamiltonian or canonical formalism: geometrical gravity theories treat space and time on the same footing, while the Hamiltonian formalism is given in terms of a set of canonical variables taking value at a given instant of time. Therefore, we need to adapt the description of the spacetime dynamics given by the geometrical theory of gravity to the Hamiltonian approach[10]. This

canonical formulation requires a spacetime splitting[11]. The line element, which carries out the splitting of the (3+1)-spacetime into foliated 3-dimensional space-like hypersurfaces, is well known in the literature[1, 5, 11, 12] and is written as

$$ds^2 = (N^i N_i - N^2)dt^2 + 2N_j dx^j dt + h_{ij} dx^i dx^j, \tag{7}$$

where the scalar function N is called the *lapse function*, N^i refers to the components of a spatial 3-vector, called *shift vector*, and h_{ij} is the induced 3-metric on Σ_t. It is simple to identify the metric components in terms of N, N^i and h_{ij}:

$$g_{00} = N^i N_i - N^2 \quad , \quad g_{0j} = N_j \quad , \quad g_{ij} = h_{ij}, \tag{8}$$

with the contravariant components

$$g^{00} = -\frac{1}{N^2} \quad , \quad g^{0j} = \frac{N^j}{N^2} \quad , \quad g^{ij} = h^{ij} - \frac{N^i N^j}{N^2} \tag{9}$$

and $\sqrt{-g} = N\sqrt{h}$. Therefore, the splitting of the spacetime is achieved mathematically by foliating the spacetime in a series of space-like hypersurfaces Σ_t, labeled by a coordinate t through a function $t(x^\mu)$ defined on the spacetime[12]. The induced metric on Σ_t can be express in four-dimensional notation by

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu. \tag{10}$$

Taking into account that $n_\mu \equiv (-N, \vec{0})$, it is straightforward to obtain $h_{00} = N^i N_i$, $h_{0i} = N_i$ and $h_{ij} = g_{ij}$. Although $h_{\mu\nu}$ is a space-like 3-metric, h_{00} and h_{0i} are non-null because $g_{0i} \neq 0$. These non-null components are required to cancel the effects of g_{0i}. Furthermore, by definition of the induced metric (10), we have $h^\mu_{\ \nu} n^\nu = 0$ as an orthogonality relation.

Besides, taking into consideration the non-metricity Weylian condition (1) and the normalization relation $n_\mu n^\mu = -1$, we can obtain

$$n^\mu \nabla_\nu n_\mu = -\frac{1}{2}\phi_\nu \quad \text{and} \quad n_\mu \nabla_\nu n^\mu = \frac{1}{2}\phi_\nu, \tag{11}$$

The relations above, result from the difference between Riemannian and Weylian geometries, and are essential to turn the geometric quantities associated to the gravitational field into functions that depend on the Weyl field. Naturally, the known results of general relativity will arise in the limit when the Weyl field vanishes.

IV. EXTRINSIC CURVATURE AND GAUSS-CODAZZI EQUATIONS

The induced metric $h_{\mu\nu}$ itself is an intrinsic quantity, and as a metric it allows us to define a unique covariant derivative operator D_μ on Σ_t, by using the projection tensor $h^\alpha_{\ \beta} = \delta^\alpha_{\ \beta} + n^\alpha n_\beta$ on the 4-dimensional covariant derivative $\nabla_\mu X_\nu$:

$$D_\alpha X_\beta \doteq h^\mu_{\ \alpha} h^\nu_{\ \beta} \nabla_\mu X_\nu. \tag{12}$$

Hence, assuming the rule $D_\alpha f = h^\beta_{\ \alpha} \nabla_\alpha f$, where f is a scalar function, we can determine the action of D_μ on the contravariants vectors,

$$D_\alpha V^\beta = h^\mu_{\ \alpha} h^\beta_{\ \nu} \nabla_\mu V^\nu. \tag{13}$$

It is important to note that the spatial covariant derivative of the induced metric also reflects the consequences of the non-metricity condition. Indeed, it is not difficult to see that

$$D_\alpha h_{\mu\nu} = h^\beta_{\ \alpha} \phi_\beta h_{\mu\nu}, \tag{14}$$

since $h^\lambda_{\ \nu} n_\lambda = 0$ and $\nabla_\beta g_{\gamma\lambda} = \phi_\beta g_{\gamma\lambda}$. Thus, if $h^\beta_{\ \alpha}\phi_\beta$ is the gradient of the Weyl field projected on Σ_t, we have the interesting case where the metricity condition holds on Σ_t. This case occurs when the Weyl field depends only on the time coordinate, i.e., $\phi_\mu \equiv \phi_\mu(t)$, which is a plausible hypothesis for homogeneous and isotropic cosmological models. Therefore, since $\phi_\mu \propto n_\alpha$ we have $h^\beta_{\ \alpha}\phi_\beta = 0$, and (14) becomes

$$D_\alpha h_{\mu\nu} = 0. \tag{15}$$

Now we are able to describe the intrinsic features of the spatial slices, since we have the induced metric and a covariant derivative intrinsic to Σ_t. However, to obtain the complete information about the structure of the spacetime, we need to know how the hypersurfaces are embedded in the spacetime. Intuitively we expect that this information is contained in the manner how Σ_t vary from point to point. This variation is given by the so-called *extrinsic curvature* of Σ_t, defined by

$$\mathcal{K}_{\alpha\beta} \doteq -h^\mu_\alpha h^\nu_\beta \nabla_\mu n_\nu. \tag{16}$$

In Riemannian terms, we have the useful expression

$$\mathcal{K}_{\alpha\beta} = \widetilde{K}_{\alpha\beta} + \frac{1}{2} h_{\alpha\beta}\xi, \tag{17}$$

where $\xi \doteq n_\mu \phi^\mu$ is the normal component of the Weyl field and $\widetilde{K}_{\alpha\beta} \doteq -h^\mu_\alpha h^\nu_\beta \widetilde{\nabla}_\mu n_\nu$ is the Riemannian extrinsic curvature. We are denoting the quantities built with the Riemannian connection with a tilde ($\widetilde{\ }$). For instance, $\widetilde{\nabla}_\mu V^\alpha \doteq \partial_\mu V^\alpha + \{^{\alpha}_{\mu\nu}\} V^\nu$ and $\widetilde{\Box} \doteq g^{\mu\nu}\widetilde{\nabla}_\mu \widetilde{\nabla}_\nu$, where the latter denotes the Laplace-Beltrami operator.

In other words, the Weylian extrinsic curvature differs from the riemannian one by the projections of the Weyl field (its gradient in fact) in the normal direction. The extrinsic curvature $\mathcal{K}_{\mu\nu}$ is a symmetric tensor[13], $\mathcal{K}_{\mu\nu} = \mathcal{K}_{\nu\mu}$, and its contravariant components are purely spatial. From equations (10) and (16), and defining the 4-vector acceleration as

$$a_\mu = n^\alpha \nabla_\alpha n_\mu, \tag{18}$$

we conclude that the 4-dimensional covariant derivative of the normal vector can be decomposed in the following terms

$$-\nabla_\mu n_\nu = \mathcal{K}_{\mu\nu} + n_\mu a_\nu - \frac{1}{2} n_\nu \phi_\alpha h^\alpha_{\ \mu}. \tag{19}$$

This equation is helpful to write the curvature tensor in terms of $\mathcal{K}_{\mu\nu}$. Moreover, it is useful to obtain the expression of the spatial components of $\mathcal{K}_{\nu\mu}$ in terms of $\{N, N^i, h_{ij}, \phi_\mu\}$. Thus, by definition, we have

$$\mathcal{K}_{ij} = \frac{1}{N}\left[D_{(i}N_{j)} - \phi_{(i}N_{j)} + \frac{1}{2}\left(h_{ij}\dot{\phi} - \dot{h}_{ij}\right)\right], \tag{20}$$

where parentheses are being used to denote symmetrization of indices

$$S_{(\mu\nu)} \equiv \frac{1}{2}\left(S_{\mu\nu} + S_{\nu\mu}\right), \tag{21}$$

and dots represent the time coordinate derivatives, i.e., $\dot{\phi} \equiv \partial_0 \phi$ and $\dot{h}_{ij} \equiv \partial_0 h_{ij}$. Thus, we note the presence of a time derivative of the Weyl field in the expression of the extrinsic curvature, which implies that the Weyl field plays the role of a canonical variable of the system. Besides, it is clear that in a particular coordinate system in which N^i is null, we would get

$$\mathcal{K}_{ij} = \frac{1}{2N}\left(h_{ij}\dot{\phi} - \dot{h}_{ij}\right). \tag{22}$$

Once a particular foliation of the Weyl spacetime is given, $h_{\mu\nu}$ and $\mathcal{K}_{ij}$ contain the information about the intrinsic and extrinsic features of the Σ_t. Thus, in order to write the 4-dimensional curvature tensor $R^\delta_{\ \rho\sigma\gamma}$ in terms of $\mathcal{K}_{\mu\nu}$ and the curvature tensor of the 3-dimensional hypersurface ${}^{(3)}R^\alpha_{\ \beta\mu\nu}$, we must use the *Gauss-Codazzi equations*:

$${}^{(3)}R_{\alpha\beta\mu\nu} = h^\delta_{\ \alpha} h^\rho_{\ \beta} h^\sigma_{\ \mu} h^\gamma_{\ \nu} R_{\delta\rho\sigma\gamma} + \mathcal{K}_{\alpha\mu}\mathcal{K}_{\beta\nu} - \mathcal{K}_{\alpha\nu}\mathcal{K}_{\beta\mu}. \tag{23}$$

Finally, the above expression (23) allows us to deduce the ADM Lagrangian corresponding to a given gravity theory. In this paper, we will be concerned with theories whose Lagrangians consists of the curvature scalar R and a kinetic term with respect to the Weyl field.

V. THE ADM LAGRANGIAN IN WEYL-TYPE THEORIES

Until now we have obtained the appropriate generalizations of some geometric quantities from a Riemannian context to the case of Weyl integrable manifold. This enables us to deduce the ADM Lagrangian for any Weyl-type gravity theory. Examples of the latter are a proposal known in the literature as the *Weyl integrable spacetime* gravity (WIST)[14, 15], described by the action

$$\mathcal{S}_{wist} = \int \sqrt{-g}\,(R + \omega \phi_\alpha \phi^\alpha)\, d^4x, \tag{24}$$

and also a geometric approach to Brans-Dicke theory (GBD)[16], which postulates the action

$$\mathcal{S}_{GBD} = \int \sqrt{-g}\mathrm{e}^{-\phi}\,(R + \omega \phi_\alpha \phi^\alpha)\, d^4x, \tag{25}$$

where R is the Weyl curvature scalar, ϕ is the Weyl field and ω is dimensionless parameter. In order to obtain R in terms of the extrinsic and intrinsic curvatures we contract $R_{\lambda\nu\sigma\tau}$ with $h_{\alpha\beta}$, which gives

$$R = h^{\lambda\sigma} h^{\nu\tau} R_{\lambda\nu\sigma\tau} - 2n^\lambda n^\sigma R_{\lambda\sigma}. \tag{26}$$

It can be shown that

$$R = {}^{(3)}R - \mathcal{K}^\gamma_{\ \nu}\mathcal{K}^\nu_{\ \gamma} + \mathcal{K}^2 + 2\left(\nabla_\mu + \phi_\mu\right) A^\mu, \tag{27}$$

where we define $A^\mu \doteq a^\mu + n^\mu(\mathcal{K} - \frac{1}{2}\xi)$, $\mathcal{K}$ is the trace of $\mathcal{K}_{\mu\nu}$ and ${}^{(3)}R$ represents the 3-dimensional curvature scalar, i.e., the curvature scalar calculated with the 3-metric h_{ij}. On the other hand, we have

$$\nabla_\alpha A^\alpha = \widetilde{\nabla}_\alpha A^\alpha - \frac{1}{2}\left(\delta^\alpha_{\ \alpha}\phi_\beta + \delta^\alpha_{\ \beta}\phi_\alpha - g_{\alpha\beta}\phi^\alpha\right) A^\beta, \tag{28}$$

which leads to

$$\widetilde{\nabla}_\alpha A^\alpha \equiv (\nabla_\alpha + 2\phi_\alpha)\, A^\alpha. \tag{29}$$

In this way, we can write

$$R = L_{ADMW} + 2\left(\nabla_\mu + 2\phi_\mu\right) A^\mu \tag{30}$$

where the latter term in (30) is the *Riemannian divergence* shown in (29). Finally the ADM Lagrangian in Weyl integrable spacetime will be given by

$$L_{ADMW} \doteq {}^{(3)}R - \mathcal{K}^{\gamma}_{\ \nu}\mathcal{K}^{\nu}_{\ \gamma} + \mathcal{K}^2 - 2\phi_{\mu}A^{\mu}. \tag{31}$$

Another way to obtain an expression of the ADM Lagrangian in a Weyl integrable spacetime is by decomposing R in terms of Riemannian quantities and ϕ.[9]. Indeed, from

$$R_{\mu\nu} \equiv \widetilde{R}_{\mu\nu} - \frac{3}{2}\widetilde{\nabla}_{\nu}\phi_{\mu} + \frac{1}{2}\widetilde{\nabla}_{\mu}\phi_{\nu} - \frac{1}{2}\phi_{\mu}\phi_{\nu} + \frac{1}{2}g_{\mu\nu}\left(\phi^{\alpha}\phi_{\alpha} - \widetilde{\nabla}_{\alpha}\phi^{\alpha}\right), \tag{32}$$

we obtain,

$$R = \widetilde{R} - 3\widetilde{\Box}\phi + \frac{3}{2}g^{\mu\nu}\phi_{\mu}\phi_{\nu}. \tag{33}$$

Therefore, given an arbitrary Weyl-like action

$$\mathcal{S}_G = \int \sqrt{-g}f(\phi)\left[R + j(\phi)g^{\mu\nu}\phi_{\mu}\phi_{\nu}\right]d^4x, \tag{34}$$

where $f(\phi)$ and $j(\phi)$ are functions that "label" a specific theory, we can write (34) in the form

$$\mathcal{S}_G = \int \sqrt{-g}f(\phi)\left[\widetilde{\mathcal{L}}_{ADMR} + \lambda(\phi)g^{\mu\nu}\phi_{\mu}\phi_{\nu}\right]d^4x - 2\int \sqrt{-g}f(\phi)\left[\widetilde{\Box}\phi + \widetilde{\nabla}_{\alpha}\left(\xi n^{\alpha}\right)\right]d^4x, \tag{35}$$

where we have redefined the parameter to $\lambda(\phi) \doteq \frac{3}{2} + j(\phi)$, and used that[11],

$$\widetilde{R} \equiv \widetilde{\mathcal{L}}_{ADMR} + 2\widetilde{\nabla}_{\mu}\left(\widetilde{a}^{\mu} + n^{\mu}\widetilde{K}\right). \tag{36}$$

The Riemannian ADM Lagrangian $\widetilde{\mathcal{L}}_{ADMR}$ is well known in the literature[1, 11, 12], and is given by

$$\widetilde{\mathcal{L}}_{ADMR} \doteq {}^{(3)}\widetilde{R} - \widetilde{K}^{\mu}_{\ \nu}\widetilde{K}^{\nu}_{\ \mu} + \widetilde{K}^2. \tag{37}$$

For simplicity, we group the terms with divergences in the definition of the function D_{iv}:

$$D_{iv} \doteq \int \sqrt{-g}f(\phi)\left[\widetilde{\Box}\phi + \widetilde{\nabla}_{\alpha}\left(\xi n^{\alpha}\right)\right]d^4x, \tag{38}$$

where in this (3+1) foliation, we have

$$\xi \doteq g^{\mu\nu}n_{\mu}\phi_{\nu} \equiv \frac{\dot{\phi}}{N^2}. \tag{39}$$

We can work out the surface terms that appear in (38) and thus write

$$\mathcal{S}_G = \int \sqrt{-g}f(\phi)\left[\widetilde{\mathcal{L}}_{ADMR} + \lambda\phi_{\mu}\phi^{\mu}\right]d^4x + 2F_{\phi} + Surface\ Terms, \tag{40}$$

with

$$F_\phi \doteq \int \sqrt{-g} \left[f' \left(\frac{\dot{\phi}}{N^2} + \phi \widetilde{\Box} \phi \right) + \phi f'' \phi_\mu \phi^\mu \right] d^4x, \tag{41}$$

and $f' \doteq \frac{\partial f}{\partial \phi}$, $f'' \doteq \frac{\partial^2 f}{\partial \phi^2}$.

In this way we have shown that the Weyl field is identified with a canonical variable, since there is canonical momentum conjugated to it. The functional form of this momentum depends on $f(\phi)$ and $j(\phi)$. In the case of WIST, where $f(\phi) = \text{const.}$ and $\lambda = \omega$, we get

$$\mathcal{S}_G = \int \sqrt{-g} \left(\widetilde{\mathcal{L}}_{ADMR} + \omega g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right) d^4x. \tag{42}$$

Let us note that the expression for the action (42) is identical (with $\omega = -1/2$) to the case of quantum cosmology in the context of general relativity minimally coupled to a massless scalar field in a FLRW cosmological models[17]. We are now ready to proceed to the quantization of FLRW cosmological models determined by the Weyl integrable spacetime gravity theory. We leave this for future work.

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